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Applied Mathematics Letters 17 (2004) 631–639
Applied Mathematics Letters
www.elsevier.com/locate/aml

Frame Wavelets with Matrix Dilations in $L^2(\mathbf{R}^n)$

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(Received and accepted November 2003)

Communicated by P. L. Butzer

Abstract—In this paper, we obtain a necessary condition and a sufficient condition for a general wavelet with a matrix dilation to be a frame in $L^2(\mathbf{R}^n)$. We extend the concept of frame wavelet sets and give their constructions. Several examples are presented and compared with some known results.

Keywords—Wavelet, Frame, Band-limited function, Expansive matrix.

1. INTRODUCTION

A family $\{f_\lambda\}_{\lambda\in\Lambda}$ of elements in a Hilbert space $\mathcal{H}$ is a frame for $\mathcal{H}$, where Λ is a countable set, if there exist constants $L,M > 0$ such that

$$L\|f\|^2 \le \sum_{\lambda\in\Lambda} |\langle f, f_\lambda\rangle|^2 \le M\|f\|^2, \qquad \text{for any } f \in \mathcal{H}.$$

Let n be a positive integer. An $n \times n$ real matrix A is said to be expansive if all of its eigenvalues have absolute values greater than 1. In this paper, A^* denotes the transpose of A. For $\psi \in L^2(\mathbf{R}^n)$, its Fourier transform is defined by $\hat{\psi}(\xi) = \int_{\mathbf{R}^n} \psi(x)e^{-2\pi i x\cdot\xi}\,dx$, $\xi \in \mathbf{R}^n$.

For $\psi \in L^2(\mathbf{R})$, $a > 1$, and $b > 0$, let $\psi_{j,k}(x) := a^{j/2}\psi(a^jx - kb)$, $j, k \in \mathbf{Z}$.

Christensen [1] gave a sufficient condition for $\{\psi_{j,k}\}_{j,k}$ being a frame for $L^2(\mathbf{R})$.

This work was supported by the National Natural Science Foundation of China under Grant 10171050 and the Research Fund for the Doctoral Program of Higher Education.
The authors would like to thank the referees for their useful suggestions and comments.

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.aml.2003.11.001

PROPOSITION 1.1. *Let $a > 1$, $b > 0$, and $\psi \in L^2(\mathbf{R})$ be given. Assume the following.*

(i)

$$L := \frac{1}{b} \inf_{|\xi| \in [1,a]} \left(\sum_{j\in\mathbf{Z}} \left| \hat{\psi}\left(a^j \xi\right) \right|^2 - \sum_{k\neq 0} \sum_{j\in\mathbf{Z}} \left| \hat{\psi}\left(a^j\xi\right) \hat{\psi}\left(a^j\xi + \frac{k}{b}\right) \right| \right) > 0.$$

(ii)

$$M := \frac{1}{b} \sup_{|\xi|\in[1,a]} \sum_{j,k\in\mathbf{Z}} \left| \hat{\psi}\left(a^j\xi\right) \hat{\psi}\left(a^j\xi + \frac{k}{b}\right) \right| < \infty.$$

Then, $\{\psi_{j,k}\}_{j,k}$ is a frame for $L^2(\mathbf{R})$ with bounds L,M.

Chui and Shi [2] proved that if $\{\psi_{j,k}\}_{j,k}$ is a frame for $L^2(\mathbf{R})$ with bounds L, M, then

$$bL \leq \sum_{j\in\mathbf{Z}} \left| \hat{\psi}\left(a^j\xi\right) \right|^2 \leq bM, \text{ a.e.}$$

Are there similar results in higher-dimensional cases? By Chui and Shi [3], the following necessary condition holds.

PROPOSITION 1.2. *If $\psi \in L^2(\mathbf{R}^n)$, $b > 0$, $A = \lambda U$, where $\lambda > 1$, U is an unitary matrix, and $\{|\det A|^{j/2}\psi(A^j x - bk)\}_{j\in\mathbf{Z},\, k\in\mathbf{Z}^n}$ is a frame for $L^2(\mathbf{R}^n)$ with bounds L, M, then*

$$bL \leq \sum_{j\in\mathbf{Z}} \left| \hat{\psi}\left(A^{*^j}\xi\right) \right|^2 \leq bM, \qquad \text{a.e. } \xi \in \mathbf{R}^n.$$

In this paper, for a real expansive matrix A and a real nonsingular matrix B, we extend Propositions 1.1 and 1.2 in the following Theorem 1.2, in which we give a necessary condition and a sufficient condition for $\{|\det A|^{j/2}\psi(A^j x - Bk)\}_{j\in\mathbf{Z},\, k\in\mathbf{Z}^n}$ being a frame for $L^2(\mathbf{R}^n)$.

On the other hand, let A be a real expansive matrix, $E \subset \mathbf{R}^n$ be a measurable set and $\hat{\psi} = \chi_E$, Dai *et al.* [4,5] gave a necessary condition and a sufficient condition for the system $\{|\det A|^{1/2}\psi(A^j x - k)\}_{j\in\mathbf{Z},\, k\in\mathbf{Z}^n}$ being a frame for $L^2(\mathbf{R}^n)$.

In this paper, based on Theorem 2.1, we study a similar problem in a different form. Let A be a fixed real expansive matrix. A measurable set $E \subset \mathbf{R}^n$ is said to be a *frame-set* with respect to A, if there exists a real nonsingular matrix B such that $\{|\det A|^{1/2}\psi(A^j x - Bk)\}_{j\in\mathbf{Z},\, k\in\mathbf{Z}^n}$ is a frame for $L^2(\mathbf{R}^n)$, where $\hat{\psi} = \chi_E$. We give a sufficient and necessary condition for E to be a frame-set when E is bounded. In the last section, several examples are presented and compared with some known results.

2. DEFINITIONS AND MAIN RESULTS

In what follows, we assume that $\psi \in L^2(\mathbf{R}^n)$, A is a real expansive matrix, B is a $n \times n$ real nonsingular matrix, and let $\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - Bk)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^n$, $x \in \mathbf{R}^n$.

When $\{\psi_{j,k}\}_{j,k}$ is a frame for $L^2(\mathbf{R}^n)$, ψ is said to be a frame wavelet for the matrices A and B. For a nonsingular matrix B, let $\|B\| = \sup_{x\in\mathbf{R}^n} |Bx|/|x|$, where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, for any $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Let $\mu(\cdot)$ be the Lebesgue measure on $\mathbf{R}^n$. For E,$F \subset \mathbf{R}^n$, $E \overset{\text{a.e}}{=} F$ means that $\mu(E \setminus F) = \mu(F \setminus E) = 0$.

DEFINITION 2.1. *Let A be an expansive matrix. A set S in $\mathbf{R}^n$ is called a basic set of $\mathbf{R}^n$ with respect to A if S can be expressed by*

$$S = \Gamma \Bigg\backslash \bigcup_{j=-\infty}^{-1} \left(A^{*^j}\Gamma\right),$$

where Γ is a bounded measurable subset in $\mathbf{R}^n$ and the origin is an interior point of Γ.

Now, we give a necessary condition and a sufficient condition for a general wavelet with a matrix dilation to be a frame for $L^2(\mathbf{R}^n)$.

THEOREM 2.1. *Suppose that A is an expansive matrix and B is a nonsingular matrix. Then,*

(i) *if $\{\psi_{j,k}\}_{j\in\mathbf{Z},\,k\in\mathbf{Z}^n}$ is a frame for $L^2(\mathbf{R}^n)$ with bounds L,M, then*

$$L \le \frac{1}{|\det B|}\sum_{j\in\mathbf{Z}}\left|\hat{\psi}\left(A^{*^j}\xi\right)\right|^2 \le M, \qquad \text{a.e. } \xi\in\mathbf{R}^n. \tag{2.1}$$

(ii) *let S be a basic set with respect to A, if*

$$\delta := \frac{1}{|\det B|}\inf_{\xi\in S}\left(\sum_{j\in\mathbf{Z}}\left|\hat{\psi}\left(A^{*^j}\xi\right)\right|^2 - \sum_{j\in\mathbf{Z}}\sum_{l\in\mathbf{Z}^n\setminus\{0\}}\left|\hat{\psi}\left(A^{*^j}\xi\right)\hat{\psi}\left(A^{*^j}\xi + B^{*^{-1}}l\right)\right|\right) > 0 \tag{2.2}$$

and

$$\Delta := \frac{1}{|\det B|}\sup_{\xi\in S}\sum_{j\in\mathbf{Z},\,l\in\mathbf{Z}^n}\left|\hat{\psi}\left(A^{*^j}\xi\right)\hat{\psi}\left(A^{*^j}\xi + B^{*^{-1}}l\right)\right| < \infty, \tag{2.3}$$

then $\{\psi_{j,k}\}_{j,k}$ is a frame for $L^2(\mathbf{R}^n)$ with bounds δ, Δ.

COROLLARY 2.2. *If (2.1) holds and ψ is band-limited, then $\{\psi_{j,k}\}$ is a frame for any nonsingular matrix B with $1/\|B\| > \operatorname{diam}(\operatorname{supp}\hat{\psi})$.*

DEFINITION 2.2. *A family of sets $\{E_j\}_{j\in\Lambda}$ is called a finitely disjoint partition of a set E, where $\Lambda\subset\mathbf{Z}$, if the following two conditions are satisfied.*

(i) *$E = \bigcup_{j\in\Lambda}E_j$.*
(ii) *There exists an integer $N\ge 2$, such that the intersection of any N sets in $\{E_j\}_{j\in\Lambda}$ has measure zero.*

On constructions of bounded frame-sets, we have the following.

THEOREM 2.3. *Let A be an expansive matrix, $E\subset\mathbf{R}^n$ be a bounded measurable set with $0<\mu(E)<\infty$. Then, the following holds.*

(i) *If E is a frame-set with respect to A, then for any basic set S with respect to A, there exists an integer J, such that*

$$E \overset{\text{a.e}}{=} \bigcup_{j=-\infty}^{J} A^{*^j}S_j, \tag{2.4}$$

where $\{S_j\}_{j\le J}$ is a finitely disjoint partition of S.

(ii) *If there exists a basic set S with respect to A, such that (2.4) holds, then E is a bounded frame-set with respect to A.*

3. PROOFS OF MAIN RESULTS

The following proposition can be found in [6–8].

PROPOSITION 3.1. *If A is an expansive matrix, then the following holds.*

(i) *A^* is also an expansive matrix.*
(ii) *There exist constants $C>0$ and $\lambda>1$, such that for any integer $j>0$ and any $\xi\in\mathbf{R}^n$,*

$$\left|A^{-j}\xi\right| \le \frac{1}{C\lambda^j}|\xi|, \qquad \left|A^j\xi\right| \ge C\lambda^j|\xi|.$$

By Proposition 3.1, we have the following.

LEMMA 3.2. *Assume that $S \subset \mathbf{R}^n$ is a bounded measurable set, A is an expansive matrix. Then, S is a basic set with respect to A if and only if the following two items holds.*

(i) *The origin is an interior point of $\mathbf{R}^n \setminus S$.*

(ii) *Any two sets in $\{A^{*^j} S\}_{j\in\mathbf{Z}}$ are disjoint, and $\mathbf{R}^n \setminus \{0\} = \bigcup_{j\in\mathbf{Z}} A^{*^j} S$.*

PROOF. Let $D = A^*$. By Proposition 3.1, D is an expansive matrix and there exist constants $C > 0$, $\lambda > 1$ such that for any integer $j > 0$ and any $\xi \in \mathbf{R}^n$,

$$\left|D^{-j}\xi\right| \le \frac{1}{C\lambda^j}|\xi|, \qquad \left|D^j\xi\right| \ge C\lambda^j|\xi|. \tag{3.1}$$

NECESSITY. Let $S = \Gamma \setminus \bigcup_{j\le -1}(D^j\Gamma)$, where Γ is a bounded measurable subset in $\mathbf{R}^n$ and the origin is an interior point of Γ.

Since the origin is an interior point of Γ, there exists a positive constant M, such that $\{\xi : |\xi| < M\} \subset \Gamma$. Then,

$$\left\{D\eta : |\eta| < \frac{M}{\|D\|}\right\} \subset \{\xi : |\xi| < M\} \subset \Gamma. \tag{3.2}$$

By (3.2) and the expression of S, we have that $\{\eta : |\eta| < M/\|D\|\} \subset D^{-1}\Gamma \subset \mathbf{R}^n \setminus S$. Thus, we get (i).

Secondly, by the expression of S, we see that any two sets in $\{D^jS\}_{j\in\mathbf{Z}}$ are disjoint. Moreover, fix any $\xi \in \mathbf{R}^n \setminus \{0\}$, since the origin is an interior point of Γ, it follows from (3.1) that there exists an positive integer J_ξ such that $D^{-j}\xi \in \Gamma$, for any $j \ge J_\xi$. Thus,

$$\xi \in D^j\Gamma, \qquad \text{for any } j \ge J_\xi. \tag{3.3}$$

On the other hand, since Γ is bounded, we have from (3.1) that there exists a negative integer j_ξ, such that

$$\xi \notin D^j\Gamma, \qquad \text{for any } j < j_\xi. \tag{3.4}$$

It follows from (3.3) and (3.4), that there exists $p \in \mathbf{Z}$, such that $p = \min\{j : \xi \in D^j\Gamma,\ j \in \mathbf{Z}\}$. Then, $\xi \in D^p\Gamma \setminus \bigcup_{j\le p-1}(D^j\Gamma) = D^pS$. Thus, $\mathbf{R}^n \setminus \{0\} \subset \bigcup_{j\in\mathbf{Z}} D^jS$. But $0 \notin S$ by (i). Then, $0 \notin D^jS$, for any $j \in \mathbf{Z}$. Therefore, $\mathbf{R}^n \setminus \{0\} = \bigcup_{j\in\mathbf{Z}} D^jS$. Thus, we get (ii).

SUFFICIENCY. Suppose that S is a measurable set with Properties (i) and (ii).

Let $\Gamma = \{0\} \cup \bigcup_{j\le 0} D^jS$. By (ii), we have that $S = \Gamma \setminus \bigcup_{j\le -1}(D^j\Gamma)$. We are to show that Γ is bounded and the origin is an interior point of Γ.

In fact, since S is bounded, $\bigcup_{j\le 0} D^jS$ is bounded by (3.1). Thus, Γ is bounded. Secondly, by (i), there exists a positive constant M such that $|\xi| > M$, for any $\xi \in S$. Then, it follows from (3.1) that for any $\xi \in S$ and any integer $j > 0$, we have

$$\left|D^j\xi\right| \ge C\lambda^j|\xi| > CM. \tag{3.5}$$

Then, we have from (3.5) and (ii) that

$$\{\eta : |\eta| < CM\} \subset \mathbf{R}^n \Big\backslash \bigcup_{j\ge 1} (D^jS) = \{0\} \cup \bigcup_{j\le 0} D^jS = \Gamma$$

Hence, the origin is an interior point of Γ.

This completes the proof of Lemma 3.2.

PROOF OF THEOREM 2.1 (i). Let $D = A^*$, $T = B^{*^{-1}}$. It is easy to see that

$$\hat{\psi}_{j,k}(\omega) = \int_{\mathbf{R}^n} \psi_{j,k}(x)e^{-2\pi i x\cdot\omega}\,dx = |\det D|^{-j/2}\hat{\psi}\left(D^{-j}\omega\right)e^{-2\pi i Bk\cdot D^{-j}\omega}.$$

Let $p_j = |\det D|^j / |\det B|^2$, $\mathbf{T}^n = [-1/2, 1/2]^n$. For any $f \in L^2(\mathbf{R}^n)$, since $\hat{f}, \hat{\psi} \in L^2(\mathbf{R}^n)$, we have

$$
\begin{aligned}
\sum_{k\in\mathbf{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 &= \sum_{k\in\mathbf{Z}^n} |\det D|^{-j} \left| \int_{\mathbf{R}^n} \hat{f}(\xi)\overline{\hat{\psi}(D^{-j}\xi)} e^{2\pi i Bk\cdot D^{-j}\xi} d\xi \right|^2 \\
&= \sum_{k\in\mathbf{Z}^n} |\det D|^j \left| \int_{\mathbf{R}^n} \hat{f}(D^j\xi)\, \overline{\hat{\psi}(\xi)} e^{2\pi i Bk\cdot\xi} d\xi \right|^2 \\
&= p_j \sum_{k\in\mathbf{Z}^n} \left| \int_{\mathbf{R}^n} \hat{f}(D^jT\eta)\, \overline{\hat{\psi}(T\eta)} e^{2\pi i k\cdot\eta} d\eta \right|^2 \\
&= p_j \sum_{k\in\mathbf{Z}^n} \left| \sum_{l\in\mathbf{Z}^n} \int_{\mathbf{T}^n+l} \hat{f}(D^jT\eta)\, \overline{\hat{\psi}(T\eta)} e^{2\pi i k\cdot\eta} d\eta \right|^2 \\
&= p_j \sum_{k\in\mathbf{Z}^n} \left| \int_{\mathbf{T}^n} \left(\sum_{l\in\mathbf{Z}^n} \hat{f}(D^jT(\eta+l))\, \overline{\hat{\psi}(T(\eta+l))} \right) e^{2\pi i k\cdot\eta} d\eta \right|^2 \\
&= p_j \int_{\mathbf{T}^n} \left| \sum_{k\in\mathbf{Z}^n} \hat{f}(D^jT(\eta+k))\, \overline{\hat{\psi}(T(\eta+k))} \right|^2 d\eta \\
&= p_j \int_{\mathbf{T}^n} \sum_{k,m\in\mathbf{Z}^n} \hat{f}(D^jT(\eta+k))\, \overline{\hat{\psi}(T(\eta+k)) \hat{f}(D^jT(\eta+m))} \hat{\psi}(T(\eta+m))\, d\eta \\
&= p_j \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{T}^n+k} \hat{f}(D^jT\eta)\, \overline{\hat{\psi}(T\eta)} \\
&\quad \cdot \sum_{m\in\mathbf{Z}^n} \overline{\hat{f}(D^jT(\eta+(m-k)))} \hat{\psi}(T(\eta+(m-k)))\, d\eta \\
&= p_j \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}^n} \hat{f}(D^jT\eta)\, \overline{\hat{f}(D^jT(\eta+l))\hat{\psi}(T\eta)} \hat{\psi}(T(\eta+l))\, d\eta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{j\in\mathbf{Z}} \sum_{k\in\mathbf{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 &= \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}^n} \hat{f}(\xi)\overline{\hat{f}(\xi + D^jTl)\hat{\psi}(D^{-j}\xi)} \hat{\psi}(D^{-j}\xi + Tl)\, d\xi \\
&= M(f) + R(f),
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
M(f) &= \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}(D^j\xi)\right|^2 d\xi = \frac{1}{|\det B|} \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \sum_{j\in\mathbf{Z}} \left|\hat{\psi}(D^j\xi)\right|^2 d\xi, \\
R(f) &= \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \int_{\mathbf{R}^n} \sum_{l\in\mathbf{Z}^n\setminus\{0\}} \hat{f}(\xi)\overline{\hat{f}(\xi + D^{-j}Tl)\,\hat{\psi}(D^j\xi)} \hat{\psi}(D^j\xi + Tl)\, d\xi.
\end{aligned}
$$

Now, given any fixed $\omega \in \mathbf{R}^n \setminus \{0\}$, let

$$
\hat{f}_m(\xi) = \frac{1}{\sqrt{\mu(H_m)}} \chi_{\omega+H_m}(\xi), \qquad m \in \mathbf{Z}.
$$

We are to estimate $R(f_m)$. For that, let

$$
\begin{aligned}
\Omega = \mathbf{T}^n &= \left[-\frac{1}{2}, \frac{1}{2}\right]^n, \\
H_m &= D^m(\Omega), \\
q &= \min_{m\in\mathbf{Z}}\{m : 2\Omega \subset H_m\}, \\
C_1 &= \min_{m\in\mathbf{Z}}\{m : H_m \cap T\mathbf{Z}^n \neq \{0\}\}.
\end{aligned}
$$

If $l \in \mathbf{Z}^n \setminus \{0\}$ and ξ, $\xi + D^{-j}Tl \in \omega + H_m$, then $D^{-j}Tl \in D^m(2\Omega) \subset D^{m+q}(\Omega) = H_{m+q}$, $Tl \in H_{j+m+q}$, and $j + m + q \geq C_1$. Let

$$m_1 = -m - q + C_1,$$
$$S_{j,m} = \{l : l \in \mathbf{Z}^n, Tl \in H_{j+m+q}\}, \qquad \text{for } m \in \mathbf{Z} \text{ and } j \geq m_1.$$

Now, we use the fact that the number of points of $T(\mathbf{Z}^n)$, different from the origin and contained in the set H_{j+m+q}, is smaller than a constant multiple of the volume of this set (see [7; 8, p. 183]), i.e., $\#S_{j,m} \leq C|\det D|^{j+m}$, for any $m \in \mathbf{Z}$ and any $j \geq m_1$, where $\#$ denotes the number of elements in a given set, and C is a constant depending only on q and C_1. Thus, we have

$$\begin{aligned}
|R(f_m)| &= \left| \sum_{j \geq m_1} \frac{1}{|\det B|} \int_{\mathbf{R}^n} \sum_{l \in S_{j,m}} \hat{f}_m(\xi) \overline{\hat{f}_m\left(\xi + D^{-j}Tl\right) \hat{\psi}\left(D^j\xi\right)} \hat{\psi}\left(D^j\xi + Tl\right) d\xi \right| \\
&\leq \sum_{j \geq m_1} \frac{1}{|\det B|} \sum_{l \in S_{j,m}} \left(\int_{\mathbf{R}^n} \left| \hat{f}_m(\xi) \hat{\psi}\left(D^j\xi\right) \right|^2 d\xi \right)^{1/2} \\
&\quad \cdot \left(\int_{\mathbf{R}^n} \left| \hat{f}_m\left(\xi + D^{-j}Tl\right) \hat{\psi}\left(D^j\xi + Tl\right) \right|^2 d\xi \right)^{1/2} \\
&\leq \sum_{j \geq m_1} \frac{1}{|\det B|} \sum_{l \in S_{j,m}} \int_{\mathbf{R}^n} \left| \hat{f}_m(\xi) \hat{\psi}\left(D^j\xi\right) \right|^2 d\xi \\
&\leq \sum_{j \geq m_1} \frac{C|\det D|^{j+m}}{|\det B| \mu(H_m)} \int_{\omega + H_m} \left| \hat{\psi}\left(D^j\xi\right) \right|^2 d\xi \\
&= \frac{C}{|\det B|} \sum_{j \geq m_1} \int_{D^j(\omega + H_m)} \left| \hat{\psi}(\xi) \right|^2 d\xi.
\end{aligned}$$

Further, by Proposition 3.1, there exists a constant $m_0 < \min\{0, -q + C_1\}$, such that

$$\frac{1}{2}|\omega| < |\xi| < \frac{3}{2}|\omega|, \qquad \text{for any } m < m_0 \text{ and any } \xi \in \omega + H_m.$$

Therefore, there exists a constant $C_0 > 0$, such that $|\xi| \geq C_0 \lambda^{m_1} |\omega|$, for any $m < m_0$, $j \geq m_1$ and any $\xi \in D^j(\omega + H_m)$. It is easy to see that there exists a positive integer K related to ω, such that the intersection of any K sets in $\{D^j(\omega + H_m)\}_{j \geq m_1}$ is empty for any $m < m_0$. Finally, we have

$$|R(f_m)| \leq \frac{CK}{|\det B|} \int_{|\xi| \geq C_0 \lambda^{-m-q+C_1} |\omega|} \left| \hat{\psi}(\xi) \right|^2 d\xi, \qquad \text{for any } m < m_0. \tag{3.7}$$

Now,

$$L \leq M(f_m) + R(f_m) \leq M, \qquad \text{for any } m \in \mathbf{Z}, \tag{3.8}$$

where

$$M(f_m) = \frac{1}{|\det B \mu(H_m)|} \int_{\omega + H_m} \sum_{j \in \mathbf{Z}} \left| \hat{\psi}\left(D^j\xi\right) \right|^2 d\xi. \tag{3.9}$$

Since $R(f_m) \longrightarrow 0$ as $m \to -\infty$ by (3.7), it follows from (3.8), (3.9) and [8, Theorem 6] that when $m \to -\infty$, we have

$$L \leq \frac{1}{|\det B|} \sum_{j \in \mathbf{Z}} \left| \hat{\psi}\left(D^j\xi\right) \right|^2 \leq M, \qquad \text{a.e. } \xi \in \mathbf{R}^n.$$

PROOF OF THEOREM 2.1 (ii). Using (3.6) and Cauchy-Schwarz inequality, we have

$$|R(f)| \le \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \sum_{l\in\mathbf{Z}^n\setminus\{0\}} \left(\int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(D^j\xi\right)\hat{\psi}\left(D^j\xi+Tl\right)\right| d\xi\right)^{1/2}$$

$$\cdot \left(\int_{\mathbf{R}^n} \left|\hat{f}\left(\xi+D^{-j}Tl\right)\right|^2 \left|\hat{\psi}\left(D^j\xi\right)\hat{\psi}\left(D^j\xi+Tl\right)\right| d\xi\right)^{1/2}$$

$$\le \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \left(\sum_{l\in\mathbf{Z}^n\setminus\{0\}} \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(D^j\xi\right)\hat{\psi}\left(D^j\xi+Tl\right)\right| d\xi\right)^{1/2}$$

$$\cdot \left(\sum_{l\in\mathbf{Z}^n\setminus\{0\}} \int_{\mathbf{R}^n} \left|\hat{f}\left(\xi+D^{-j}Tl\right)\right|^2 \left|\hat{\psi}\left(D^j\xi\right)\hat{\psi}\left(D^j\xi+Tl\right)\right| d\xi\right)^{1/2}$$

$$= \frac{1}{|\det B|} \sum_{j\in\mathbf{Z}} \sum_{l\in\mathbf{Z}^n\setminus\{0\}} \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 \left|\hat{\psi}\left(D^j\xi\right)\hat{\psi}\left(D^j\xi+Tl\right)\right| d\xi.$$

Thus, it follows from (3.6) and Lemma 3.2 that $\{\psi_{j,k}\}_{j,k}$ is a frame for $L^2(\mathbf{R}^n)$ with bounds δ, Δ if (2.2) and (2.3) hold.

This completes the proof of Theorem 2.1.

PROOF OF THEOREM 2.3 (i). Let $D = A^*$ and E be a bounded frame-set. Then, there exist a nonsingular matrix B and positive numbers L, M, such that $\{\psi_{j,k}\}_{j,k}$ is a frame with bounds L, M, where $\hat{\psi} = \chi_E$. Thus, (2.1) holds. Therefore, we have the following two assertions.

(P_1) $\mathbf{R}^n \overset{\text{a.e.}}{=} \bigcup_{j\in\mathbf{Z}} D^{-j}E$.

(P_2) There exists an integer $N \ge 2$ such that the intersection of any N sets in $\{D^{-j}E\}_{j\in\mathbf{Z}}$ has measure zero.

For any basic set S with respect to A, we have from (P_1) that $S \overset{\text{a.e.}}{=} \bigcup_{j\in\mathbf{Z}}\{D^{-j}E\cap S\}$.

Note that E is bounded. By Proposition 3.1 and Lemma 3.1 (i), there exists a positive integer J such that $D^{-j}E\cap S = \emptyset$, for $j > J$. Hence, $S \overset{\text{a.e.}}{=} \bigcup_{j=-\infty}^{J}\{D^{-j}E\cap S\}$. Let $S_j = D^{-j}E\cap S$, for each $j \le J$. By (P_2), the intersection of any N sets in $\{S_j\}_{j\le J}$ has measure zero. Further, by Lemma 3.2, we get

$$E \overset{\text{a.e.}}{=} E \cap \bigcup_{j\in\mathbf{Z}} D^jS = \bigcup_{j\in\mathbf{Z}}\left(E\cap D^jS\right) = \bigcup_{j\le J}\left(D^j\left(D^{-j}E\cap S\right)\right) = \bigcup_{j\le J} D^jS_j.$$

PROOF OF THEOREM 2.3 (ii). Let $D = A^*$ and S be a basic set with respect to A. Assume that $E \overset{\text{a.e.}}{=} \bigcup_{j\le J} D^jS_j$, where $\{S_j\}_{j\le J}$ is a finitely disjoint partition of S. Then, we have the following.

(P_3) There exists a positive integer $N \ge 2$, such that the intersection of any N sets in $\{S_j\}_{j\le J}$ has measure zero.

Note that $S = \bigcup_{j\le J} S_j$. Let $V = \bigcup_{j\in\mathbf{Z}} D^jE$. If we denote the complement of a set F by $\bar{F}$, then by Lemma 3.2, we have

$$0 = \mu\left(\bar{\mathbf{R}^n}\right) = \mu\left(\overline{\bigcup_{m\in\mathbf{Z}} D^m\left(\bigcup_{j\le J} S_j\right)}\right) = \mu\left(\overline{\bigcup_{m\in\mathbf{Z}}\left(\bigcup_{j\le J} D^{m-j}D^jS_j\right)}\right)$$

$$\ge \mu\left(\overline{\bigcup_{m\in\mathbf{Z}}\left(\bigcup_{j\le J} D^{m-j}E\right)}\right) \ge \mu\left(\overline{\bigcup_{m\in\mathbf{Z}}\left(\bigcup_{j\le J} V\right)}\right) = \mu\left(\bar{V}\right).$$

Hence,

(P_4) $\mathbf{R}^n \overset{\text{a.e.}}{=} \bigcup_{j\in\mathbf{Z}} D^jE$.

Assume that there exist N sets $\{D^{m_p}E\}_{1\le p\le N}$ satisfying $\mu(\bigcap_{p=1}^{N} D^{m_p}E) > 0$, where $m_1 < m_2 < \cdots < m_N$. Since

$$\bigcap_{p=1}^{N} D^{m_p}E \overset{\text{a.e.}}{=} \bigcap_{p=1}^{N}\left(\bigcup_{j\le J} D^{m_p+j}S_j\right) = \bigcup_{j_1,j_2,\dots,j_N\le J}\left(D^{m_1+j_1}S_{j_1}\cap D^{m_2+j_2}S_{j_2}\cap\cdots\cap D^{m_N+j_N}S_{j_N}\right),$$

there exist $j_p \le J$, for $p = 1,\dots,N$, such that $\mu\left(\bigcap_{p=1}^{N} D^{m_p+j_p}S_{j_p}\right) > 0$.

Since the intersection of any two sets in $\{D^jS\}_{j\in\mathbf{Z}}$ has measure zero, we have $m_p+j_p = m_q+j_q$ for $1\le p,\ q\le N$. Then, $j_p \ne j_q$ for $p\ne q$, and $\mu(\bigcap_{p=1}^{N} S_{j_p}) > 0$, which contradicts ($\mathrm{P}_3$). Thus, we have the following.

(P_5) The intersection of any N sets in $\{D^jE\}_{j\in\mathbf{Z}}$ has measure zero.

Let $\hat\psi = \chi_E$. By (P_4) and (P_5), we get

$$1 \le \sum_{j\in\mathbf{Z}} |\hat\psi(D^j\xi)|^2 \le N, \qquad \text{a.e. } \xi\in\mathbf{R}^n.$$

By Corollary 2.2, ψ is a frame wavelet for $L^2(\mathbf{R}^n)$ with A and any nonsingular matrix B with $1/\|B\| > \operatorname{diam}(E)$.

This completes the proof of Theorem 2.3.

4. EXAMPLES

EXAMPLE 4.1. Let $S = (-1/2,-1/4]\cup(1/2,1]$. By Lemma 3.2, S is a basic set with respect to $A=(2)$. Let $S_0 = S$, $S_j=\emptyset$, for each $j<0$. By Theorem 2.3, $S = \bigcup_{j\le 0} 2^jS_j$ is also a bounded frame-set with respect to $A=(2)$.

EXAMPLE 4.2. Let $A = \left(\begin{smallmatrix}2 & -1\\ 1 & 2\end{smallmatrix}\right)$. Then, $D = A^* = \left(\begin{smallmatrix}2 & 1\\ -1 & 2\end{smallmatrix}\right)$. For any positive integer m,

$$D^{-m} = \begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dfrac{1}{2+i} & 0\\ 0 & \dfrac{1}{2-i}\end{pmatrix}^m\begin{pmatrix}\dfrac{i}{2} & \dfrac{1}{2}\\ -\dfrac{i}{2} & \dfrac{1}{2}\end{pmatrix}.$$

Let $D^{-m}x = (y_1^{(m)}, y_2^{(m)})$, where $x=(x_1,x_2)\in\mathbb{R}^2$. We get

$$\max\left\{\left|y_1^{(m)}\right|,\left|y_2^{(m)}\right|\right\} \le 2\left(\frac{1}{\sqrt5}\right)^m \max\{|x_1|,|x_2|\}, \qquad \text{for any } m\ge 1.$$

Let $\Gamma = D[-1/2,1/2]^2$. Then, $D^{-m}\Gamma\subset D^{-1}\Gamma$, for any $m>1$. Thus, $S=\Gamma\setminus D^{-1}\Gamma$ is a basic set with respect to A, where Γ is a quadrangle which have four corner points $(-0.5,1.5)$, $(1.5,0.5)$, $(-1.5,-0.5)$, $(0.5,-1.5)$. By Theorem 2.3, if $\{S_j\}_{j\le J}$ is any finitely disjoint partition of S and $\hat\psi = \chi_E$, where $E=\bigcup_{j\le J} D^jS_j$ (especially $E=S$), then ψ is a frame wavelet for the matrix A and any nonsingular matrix B with $\|B\|\le 1/\operatorname{diam}(\operatorname{supp} E)$.

REMARK. For the set S in Examples 4.1 and 4.2, if B is an identity matrix and $\hat\psi=\chi_S$, then $\{\psi_{j,k}\}_{j,k}$ is not a frame (see [4, Example 5; 5, Example 1]).

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